Characterizing the Existence of Potential Functions in Weighted Congestion Games*

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Abstract

Since the pioneering paper of Rosenthal a lot of work has been done in order to determine classes of games that admit a potential. First, we study the existence of potential functions for weighted congestion games. Let C be an arbitrary set of locally bounded functions and let $\mathcal{G}(C)$ be the set of weighted congestion games with cost functions in C. We show that every weighted congestion game $G \in \mathcal{G}(C)$ admits an exact potential if and only if C contains only affine functions. We also give a similar characterization for weighted potentials with the difference that here C consists either of affine functions or of certain exponential functions. We finally extend our characterizations to weighted congestion games with facility-dependent demands and elastic demands, respectively.

1 Introduction

In many situations, the state of a system is determined by a large number of independent agents, each pursuing selfish goals optimizing an individual objective function. A natural framework for analyzing such decentralized systems are noncooperative games. It is well known that an equilibrium point in pure strategies (if it exists) need not optimize the social welfare as individual incentives are not always compatible with social objectives. Fundamental goals in algorithmic game theory are to decide whether a Nash equilibrium in pure strategies (PNE for short) exists, how efficient it is in the worst case, and how fast an algorithm (or protocol) converges to an equilibrium.

One of the most successful approaches in accomplishing these goals is the potential function approach initiated by Rosenthal [24] and generalized by Monderer and Shapley in [22]: one defines a function P on the set of possible strategies of the game and shows that every strictly improving move by one defecting player strictly reduces (increases) the value of P. Since the set of outcomes of such a game is finite, every sequence of improving moves reaches a PNE. In particular, the global minimum (maximum) of P is a PNE.

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A function P with the property above is called a *potential function* of the game. If one can associate a weight w_i to each player such that $w_i P$ decreases about the same value as the private cost of the defecting player i, then P is called a *weighted potential*. If, in addition, $w_i = 1$ for each player, then P is called an *exact potential*.

1.1 Framework

The first part of this paper studies the existence of potential functions in weighted congestion games (Definition 3.2). Congestion games, as introduced by Rosenthal [24], model the interaction of a finite set of strategic agents that compete over a finite set of facilities. A pure strategy of each player is a set of facilities. We consider cost minimization games. Here, the cost of facility f is given by a real-valued cost function c_f that depends on the number of players using f and the private cost of every player equals the sum of the costs of the facilities in the strategy that she chooses.[1] Rosenthal [24] proved in a seminal paper that such congestion games always admit a PNE by showing these games posses an exact potential function.

In a *weighted congestion game*, every player has a demand $d_i \in \mathbb{R}_{>0}$ that she places on the chosen facilities. The cost of a facility is a function of the total demand of the facility. In contrast to unweighted congestion games, weighted congestion games, even with two players, do not always admit a PNE, see the examples given by Fotakis et al. [12], Goemans et al. [15], and Libman and Orda [18].

On the positive side, Fotakis et al. [12, 13] proved that every weighted congestion game with affine cost functions possesses an exact potential function and thus, a PNE. Panagopoulou and Spirakis [23] proved existence of a weighted potential function for the case that all costs are determined by the exponential function.

The results of [12, 13] and [23] are particularly appealing as they establish existence of a potential function *independent* of the underlying game structure, that is, *independent* of the underlying strategy set, demand vector, and number of players, respectively. To further stress this independence property, we rephrase the result of Fotakis et al. as follows: Let C be a set of affine cost functions and let $\mathcal{G}(C)$ be the set of *all* weighted congestion games with cost functions in C. Then, *every* game in $\mathcal{G}(C)$ possesses an exact potential.

A natural open question is to decide whether there are further functions guaranteeing the existence of an exact or weighted potential. We thus investigate the following question: How large is the class C of (continuous) cost functions such that every game in the set of weighted congestion games $\mathcal{G}(C)$ with cost functions in C does admit a potential function and hence a PNE?

Before we outline our results we present related work and explain, why it is important to characterize weighted congestion games admitting a potential function.

1.2 Related Work

Fundamental issues in algorithmic game theory are the computability of Nash equilibria and the design of distributed dynamics (for instance best-response) that provably converge in reasonable time to a Nash equilibrium (in pure or mixed strategies).

Monderer and Shapley [22] formalized Rosenthal's approach of using potential functions to determine the existence of PNE. Furthermore, they show that one-side better response dynamics al-

[1] Since we allow the cost of a facility to be positive or negative, we also cover the maximization games.

ways converge to a PNE provided the game is finite and admits a potential. In addition, they proved that weighted potential games have other desirable properties, e.g., the Fictitious Play Process converges to a PNE [21]. For recent progress on convergence towards approximate Nash equilibria using potential functions, see Awerbuch et al. [4] and Fotakis et al. [11].

Fabrikant et al. [10] proved that one can efficiently compute a PNE for symmetric network congestion games with nondecreasing cost functions. Their proof uses a potential function argument, similar to Rosenthal [24]. Fotakis et al. [12] proved that one can compute a PNE for weighted network games with affine cost (with nonnegative coefficients) in pseudo-polynomial time (again using a potential function).

Milchtaich [20] introduced weighted congestion games with player-specific cost functions. He presented, among other results, a game on 3 parallel links with 3 players, which does not possess a PNE. On the other hand, he proved that such games with 2 players do possess a PNE. Ackermann et al. [1] characterized conditions on the strategy space in weighted congestion games that guarantee the existence of PNE. They also considered the case of player-specific cost functions.

Gairing et al. [14] derive a potential function for the case of weighted congestion games with player-specific linear latency functions (without a constant term). Mavronicolas et al. [19] prove that every unweighted congestion game with player-specific (additive or multiplicative) constants on parallel links has an ordinal potential. Even-Dar et al. [9] consider a variety of load balancing games with makespan objectives and prove among other results that games on unrelated machines possess a generalized ordinal potential function. For related results, see the survey by Vöcking [25] and references therein.

Potential functions also play a central role in Shapley cost sharing games with weighted players, which are special cases of weighted congestion games, see Anshelevich et al. [3] and Albers et al. [2]. In the variant with weighted players, each player i has a demand d_i that she wishes to place on each facility of an allowable subset of facilities (e.g., a path in a network connecting her source node s_i to her terminal node t_i). When facility $f \in F$ is stressed with a load of $\ell_f(x)$ in strategy profile x, there exists a cost of $k_f(\ell_f(x))$. Under Shapley cost sharing, this cost is shared fairly with respect to the demands among the users. Thus the cost of player i for using facility f is defined as $c_{i,f}(x) = k_f(\ell_f(x))\, d_i/\ell_f(x)$ and clearly, the private cost of player i in strategy profile x is given as $\pi_i(x) = \sum_{f \in x_i} c_{i,f}(x)$. For the unweighted case ($d_i = 1, i \in N$), Anshelevich et al. [3] proved existence of PNE and derived bounds on the price of stability using a potential function argument. This argument fails in general for games with weighted players, see the counterexamples given by Chen and Roughgarden [6]. Determining subclasses of Shapley cost sharing games with weighted players that admit a potential, however, is an open problem that we address in this paper.

1.3 Our Results for Weighted Congestion Games

Our first two results provide a characterization of the existence of exact and weighted potential functions for the set of weighted congestion games with locally bounded and continuous cost functions, respectively. Let $\mathcal{C}$ be an arbitrary set of locally bounded functions and let $\mathcal{G}(\mathcal{C})$ be the set of weighted congestion games with cost functions in $\mathcal{C}$. We show that every weighted congestion game $G \in \mathcal{G}(\mathcal{C})$ admits an exact potential if and only if $\mathcal{C}$ contains only affine functions. For an arbitrary set $\mathcal{C}$ of continuous functions, we show that every weighted congestion game $G \in \mathcal{G}(\mathcal{C})$ possesses a weighted potential if and only if exactly one of the following cases hold: (*i*) $\mathcal{C}$ contains only affine functions; (*ii*) $\mathcal{C}$ contains only exponential functions such that $c(\ell) = a_c\, e^{\phi\, \ell} + b_c$ for some

$a_c, b_c, \phi \in \mathbb{R}$, where a_c and b_c may depend on c, while ϕ must be equal for every $c \in C$.

We additionally show that the above characterizations for exact and weighted potentials are valid even if we restrict the set $\mathcal{G}(C)$ to two-player games (three-player games for weighted potentials), three-facility games (four-facility games for weighted potentials), games with symmetric strategies, games with singleton strategies, games with integral demands. Moreover, we derive a result for two-player weighted congestion games, showing that every such game with cost functions in C admits a weighted potential if $C = \{(c : \mathbb{R}_{>0} \to \mathbb{R}) : c(x) = a\, m(x) + b,\ \ a, b \in \mathbb{R}\}$, where $m : \mathbb{R}_{>0} \to \mathbb{R}$ is a strictly monotonic function.

Our results have a series of consequences. First, using a result of Monderer and Shapley [22, Lemma 2.10], our characterization of weighted potentials in weighted congestion games carries over to the mixed extension of weighted congestion games.

Second, we obtain the following characterizations for Shapley cost sharing games. Let $\mathcal{K}$ be a set of continuous functions. Then, the set $\mathcal{S}(\mathcal{K})$ of Shapley cost sharing games with weighted players and construction cost functions in $\mathcal{K}$ are weighted potential games if and only if $\mathcal{K}$ contains either quadratic construction cost functions $k(\ell) = a_k\, \ell^2 + b_k\, \ell$ or functions of type $k(\ell) = a_k\, e^{\phi\, \ell}\, \ell + b_k\, \ell$ for some $a_k, b_k, \phi \in \mathbb{R}$, where a_k and b_k may depend on k, while ϕ must be equal for every $k \in \mathcal{K}$. Notice that these results hold for arbitrary coefficients $a_k, b_k, \phi \in \mathbb{R}$. Thus, we obtain the existence of PNE for a family of games with nondecreasing and strictly concave construction costs modeling the effect of economies of scale.

After the initial publication of this paper, Harks and Klimm [16] explored the existence of PNE in weighted congestion games. For a class C of twice continuously differentiable cost functions, they showed that the conditions given in Theorem 3.9 are in fact necessary for the existence of PNE in all weighted congestion games contained in $\mathcal{G}(C)$. Their characterization, however, requires new techniques based on the analysis of generic improvement cycles, see [16] for details.

1.4 Our Results for Extended Models

In the second part of this paper, we introduce two non-trivial extensions of weighted congestion games.

First, we study weighted congestion games with *facility-dependent* demands, that is, the demand $d_{i,f}$ of player i depends on the facility f. These games contain, among others, scheduling games on identical, restricted, related and unrelated machines. In contrast to classical load balancing games, we do not consider makespan objectives. In our model, the private cost of a player is a function of the machine load multiplied with the demand of the player.

We show the following: Let C be a set of continuous functions and let $\mathcal{G}^{fd}(C)$ denote the set of weighted congestion games with facility-dependent demands. Every $G \in \mathcal{G}^{fd}(C)$ has a weighted potential if and only if C contains only affine functions. In this case the weighted potential is an exact potential. To the best of our knowledge, our characterization establishes for the first time the existence of an exact potential function (and hence the existence of a PNE) for affine cost functions and *arbitrary* strategy sets and demands, respectively.

Second, we study weighted congestion games with *elastic* demands. Here, each player i is allowed to choose both a subset of the set of facilities and her demand d_i out of a compact set $D_i \subset \mathbb{R}_{>0}$ of demands that are allowable for her. This congestion model can be interpreted as a generalization of Cournot games [8], where multiple producers strategically determine quantities they will produce. The cost of a producer is given by her offered quantity multiplied with the market price, which is usually a decreasing function of the total quantity offered by all producers. Weighted congestion games with elastic demands generalize Cournot games in the sense that there are multiple markets (facilities) and each player may offer her quantity on allowable subsets of these markets.

Weighted congestion games with elastic demands have several additional applications: they model, e.g., routing problems in the Internet, where each user wants to route data along a path in the network and adjusts the injected data rate according to the level of congestion in the network. Most mathematical models for routing and congestion control rely on fractional routing, see Kelly [17] and Cole et al. [7]. In practice, however, routing protocols use single path routing, see, e.g., the current TCP/IP protocol. Weighted congestion games with elastic demands model both congestion control and unsplittable routing. Yet another application is that of Shapley cost sharing games with players that may vary their requested demand.

Let $\mathcal{G}^e(\mathcal{C})$ be the set of weighted congestion games with elastic demands where each player may chose her demand out of a compact space and where the cost of each facility is determined by a function in $\mathcal{C}$. Our main contribution is to show that all games $G \in \mathcal{G}^e(\mathcal{C})$ are weighted potential games if and only if $\mathcal{C}$ contains only affine functions. For this important class of games, this result also establishes for the first time the existence of PNE.

2 Preliminaries

A *finite strategic game* is a tuple $G = (N, X, \pi)$ where $N = \{1, \dots, n\}$ is the non-empty finite set of players, $X = \bigtimes_{i \in N} X_i$ where X_i is the finite and non-empty set of strategies of player i, and $\pi : X \to \mathbb{R}^n$ is the combined private cost function.

We will call an element $x \in X$ strategy profile. For $S \subset N$, $-S$ denotes the complementary set of S, and we define for convenience of notation $X_S = \bigtimes_{j \in S} X_j$. Instead of $X_{-\{i\}}$ we will write X_{-i}, and with a slight abuse of notation we will write sometimes a strategy profile as $x = (x_i, x_{-i})$ meaning that $x_i \in X_i$ and $x_{-i} \in X_{-i}$.

The following definition is due to Monderer and Shapley [22].

Definition 2.1 (Weighted and exact potential games)**.** A strategic game $G = (N, X, \pi)$ is called *weighted potential game* if there is a vector $w = (w_i)_{i \in N} \in \mathbb{R}^n_{>0}$ and a function $P : X \to \mathbb{R}$ such that $\pi_i(x_i, x_{-i}) - \pi_i(y_i, x_{-i}) = w_i \left(P(x_i, x_{-i}) - P(y_i, x_{-i})\right)$ for all $i \in N$, $x_{-i} \in X_{-i}$, and all $x_i, y_i \in X_i$. The function P together with the vector w is then called a weighted potential of the game G. The function P is called an *exact* potential if $w_i = 1$ for all $i \in N$.

We sometimes call a weighted potential function P a $(w_i)_{i \in N}$-potential. Monderer and Shapley [22, Theorem 2.8] characterized exact potentials in a very convenient way. For this, let a finite strategic game $G = (N, X, \pi)$ be given. A *path* in X is a sequence $\gamma = (x^0, x^1, \dots x^m)$ with $x^k \in X$, $k = 0, \dots, m$, such that for all $k \in \{1, \dots, m\}$ there exists a unique player $i_k \in N$ such that $x^k = (x^k_{i_k}, x^{k-1}_{-i_k})$ for some $x^k_{i_k} \neq x^{k-1}_{i_k}$, $x^k_{i_k} \in X_i$. A path is called closed if $x^0 = x^m$ and is called simple if $x^k \neq x^l$ for $k \neq l$. The length of a closed path is defined as the number of its distinct elements. For a set of strategy profiles X let $\Gamma(X)$ denote the set of all simple closed paths in X that have length 4. For a finite path $\gamma = (x^0, x^1, \dots, x^m)$ let the discrete path integral of π along γ be defined as $I(\gamma, \pi) = \sum_{k=1}^m \left(\pi_{i_k}(x^k) - \pi_{i_k}(x^{k-1})\right)$ where i_k is the deviator at step k in γ, that is $x^k_{i_k} \neq x^{k-1}_{i_k}$.

Theorem 2.2 (Monderer and Shapley)**.** *Let $G = (N, X, \pi)$ be a finite strategic game. Then, G is an exact potential game if and only if $I(\gamma, \pi) = 0$ for all $\gamma \in \Gamma(X)$.*

In the following, we will use this characterization in order to study the existence of potentials in weighted congestion games.

3 Weighted Congestion Games

Definition 3.1 (Congestion model)**.** A tuple $\mathcal{M} = (N, F, X = \bigtimes_{i\in N} X_i, (c_f)_{f\in F})$ is called a *congestion model*, where $N = \{1, \ldots, n\}$ is a non-empty, finite set of players, F is a non-empty, finite set of facilities, for each player $i \in N$, her collection of pure strategies X_i is a non-empty, finite set of subsets of F and $(c_f)_{f\in F}$ is a set of cost functions.

In the following, we will define weighted congestion games similar to Goemans et al. [15].

Definition 3.2 (Weighted congestion game)**.** Let $\mathcal{M} = (N, F, X, (c_f)_{f\in F})$ be a congestion model and $(d_i)_{i\in N} \in \mathbb{R}^n_{>0}$ be a vector of demands. The corresponding *weighted congestion game* is the strategic game $G(\mathcal{M}) = (N, X, \pi)$, where π is defined as $\pi = \bigtimes_{i\in N} \pi_i$, $\pi_i(x) = \sum_{f\in x_i} d_i\, c_f(\ell_f(x))$ and $\ell_f(x) = \sum_{j\in N: f\in x_j} d_j$.

We call $\ell_f(x)$ the *load* on facility f in strategy x. In case there is no confusion on the underlying congestion model, we will write G instead of $G(\mathcal{M})$.

A slightly different class of games has been considered by (among others) Fotakis et al. [12, 13], Gairing et al. [14] and Mavronicolas et al. [19]. They considered games that almost coincide with Definition 3.2 except that the private cost of every player is not scaled by her demands. We call such games *normalized* if they comply with Definition 3.2 except that the private costs are defined as $\overline{\pi}_i(x) = \sum_{f\in x_i} c_f(\ell_f(x))$ for all $i \in N$.

Fotakis et al. [12] show that there are normalized weighted congestion games with $c_f(\ell) = \ell$ for all $f \in F$ that are not exact potential games. They also show that any normalized weighted congestion game with linear costs on the facilities admits a weighted potential.

We state the following trivial relations between weighted congestion games and normalized weighted congestion games: Let $G = (N, X, \pi)$ and $\overline{G} = (N, X, \overline{\pi})$ be a weighted congestion game and a normalized weighted congestion game with demands $(d_i)_{i\in N}$, respectively. Moreover, let them share the same congestion model and the same demands. Then G and $\overline{G}$ coincide in the following sense: (*i*) A strategy profile $x \in X$ is a PNE in G if and only if x is a PNE in $\overline{G}$; (*ii*) A real-valued function $P : X \to \mathbb{R}$ is a $(w_i/d_i)_{i\in N}$-potential for G if and only if P is a $(w_i)_{i\in N}$-potential for $\overline{G}$; (*iii*) A real-valued function $P : X \to \mathbb{R}$ is an ordinal potential for G (see [22] for a definition) if and only if P is an ordinal potential for $\overline{G}$; (*iv*) The real-valued function $P : X \to \mathbb{R}$ is an exact potential for G if and only if P is a $(d_i)_{i\in N}$-potential for $\overline{G}$; (*v*) The real-valued function $P : X \to \mathbb{R}$ is an exact potential for $\overline{G}$ if and only if P is a $(1/d_i)_{i\in N}$-potential for G. All proofs rely on the simple observation that $\pi_i(x) = d_i\, \overline{\pi}_i(x)$ for all $i \in N$, $x \in X$.

3.1 Characterizing the Existence of an Exact Potential

In the following, we will examine necessary and sufficient conditions for a weighted congestion game G to be a potential game. The criterion in Theorem 2.2 states that the existence of an exact potential for $G = (N, X, \pi)$ is equivalent to the fact that $I(\gamma, \pi) = 0$ for all $\gamma \in \Gamma(X)$. In such paths, either one or two players deviate. It is easy to verify that $I(\gamma, \pi) = 0$ for all paths γ with only one deviating player. Considering a path γ with two deviating players, say i and j, each of them uses two different strategies, say $x_i, y_i \in X_i$ and $x_j, y_j \in X_j$. We denote by $z_{-\{i,j\}} \in X_{-\{i,j\}}$ the strategy profile of all players except i and j that remains constant in γ. Then, a generic path $\gamma \in \Gamma(X)$ can be written as $\gamma = ((x_i, x_j, z_{-\{i,j\}}), (y_i, x_j, z_{-\{i,j\}}), (y_i, y_j, z_{-\{i,j\}}), (x_i, y_j, z_{-\{i,j\}}), (x_i, x_j, z_{-\{i,j\}}))$. The following lemma provides an explicit formula for the calculation of $I(\gamma, \pi)$ for such a path.

Lemma 3.3. *Let $\mathcal{M} = (N, F, X, (c_f)_{f\in F})$ be a congestion model and $G(\mathcal{M})$ a corresponding weighted congestion game with demands $(d_i)_{i\in N}$. Moreover, let*

$$\gamma = ((x_i, x_j, z_{-\{i,j\}}), (y_i, x_j, z_{-\{i,j\}}), (y_i, y_j, z_{-\{i,j\}})(x_i, y_j, z_{-\{i,j\}}), (x_i, x_j, z_{-\{i,j\}}))$$

be an arbitrary path in $\Gamma(X)$ with two deviating players. Then,

$$\begin{aligned} I(\gamma,\pi) = &\sum_{f\in F_1\cup F_{11}} (d_j - d_i)c_f(d_i + d_j + r_f) - d_j c_f(d_j + r_f) + d_i c_f(d_i + r_f) \\ &+ \sum_{f\in F_3\cup F_9} (d_i - d_j)c_f(d_i + d_j + r_f) - d_i c_f(d_i + r_f) + d_j c_f(d_j + r_f), \end{aligned} \tag{1}$$

where $F_1 = (x_i \backslash y_i) \cap (x_j \backslash y_j)$, $F_3 = (x_i \backslash y_i) \cap (y_j \backslash x_j)$, $F_9 = (y_i \backslash x_i) \cap (x_j \backslash y_j)$, and $F_{11} = (y_i \backslash x_i) \cap (y_j \backslash x_j)$.

Proof. We fix $i, j \in N$, $x_i, y_i \in X_i$, $x_j, y_j \in X_j$, and $z_{-\{i,j\}} \in X_{-\{i,j\}}$ arbitrarily and consider the path $\gamma = ((x_i, x_j, z_{-\{i,j\}}), (y_i, x_j, z_{-\{i,j\}}), (y_i, y_j, z_{-\{i,j\}})(x_i, y_j, z_{-\{i,j\}}), (x_i, x_j, z_{-\{i,j\}}))$. We compute straightforwardly that

$$\begin{aligned} I(\gamma,\pi) = &\pi_i(y_i, x_j, z_{-\{i,j\}}) - \pi_i(x_i, x_j, z_{-\{i,j\}}) + \pi_j(y_i, y_j, z_{-\{i,j\}}) - \pi_j(y_i, x_j, z_{-\{i,j\}}) \\ &+ \pi_i(x_i, y_j, z_{-\{i,j\}}) - \pi_i(y_i, y_j, z_{-\{i,j\}}) + \pi_j(x_i, x_j, z_{-\{i,j\}}) - \pi_j(x_i, y_j, z_{-\{i,j\}}). \end{aligned} \tag{2}$$

For a facility $f \in F$, we define $r_f = \sum_{m\in N\backslash\{i,j\}: f\in (z_{-\{i,j\}})_m} d_m$ as the sum of the demands on f in the partial strategy profile $z_{-\{i,j\}}$. For fixed x_i, y_i, x_j and y_j, every facility $f \in F$ can be chosen by player i in both strategy x_i and strategy y_i, in one of these strategies or not at all. The same holds for player j and strategies x_j and y_j. We can thus decompose F into 16 disjoint sets $F_1, \ldots, F_{16}$. The first set, F_1, comprises all facilities that are in $(x_i \setminus y_i) \cap (x_j \setminus y_j)$. F_2 contains all facilities that are in $(x_i \setminus y_i) \cap (x_j \cap y_j)$, and so on. The comprehensive description of all 16 cases is given in Table 1.

	$x_j \setminus y_j$	$x_j \cap y_j$	$y_j \setminus x_j$	$F \setminus (x_j \cup y_j)$
$x_i \setminus y_i$	F_1	F_2	F_3	F_4
$x_i \cap y_i$	F_5	F_6	F_7	F_8
$y_i \setminus x_i$	F_9	F_{10}	F_{11}	F_{12}
$F \setminus (x_i \cup y_i)$	F_{13}	F_{14}	F_{15}	F_{16}

Table 1: Decomposition of F into 16 disjoint subsets $F_k, k = 1, \ldots, 16$.

In order to compute for instance the first term of equation (2), we notice that in strategy profile $x = (y_i, x_j, z_{-\{i,j\}})$ the load on each facility $f \in F_5 \cup F_6 \cup F_9 \cup F_{10}$ equals $\ell_f(x) = d_i + d_j + r_f$, while the load on each facility $g \in F_7 \cup F_8 \cup F_{11} \cup F_{12}$ equals $\ell_g(x) = d_i + r_g$. These considerations lead to the following equation. We will use the notation $\sum_{F,G}$ for $\sum_{f \in F \cup G}$.

$$
\begin{aligned}
&I(\gamma,\pi) = \\
&\quad d_i \left(\sum_{F_9,F_{10}} c_f(d_i+d_j+r_f) + \sum_{F_{11},F_{12}} c_f(d_i+r_f) - \sum_{F_1,F_2} c_f(d_i+d_j+r_f) - \sum_{F_3,F_4} c_f(d_i+r_f) \right) \\
&\quad + d_j \left(\sum_{F_7,F_{11}} c_f(d_i+d_j+r_f) + \sum_{F_3,F_{15}} c_f(d_j+r_f) - \sum_{F_5,F_9} c_f(d_i+d_j+r_f) - \sum_{F_1,F_{13}} c_f(d_j+r_f) \right) \\
&\quad + d_i \left(\sum_{F_2,F_3} c_f(d_i+d_j+r_f) + \sum_{F_1,F_4} c_f(d_i+r_f) - \sum_{F_{10},F_{11}} c_f(d_i+d_j+r_f) - \sum_{F_9,F_{12}} c_f(d_i+r_f) \right) \\
&\quad + d_j \left(\sum_{F_1,F_5} c_f(d_i+d_j+r_f) + \sum_{F_9,F_{13}} c_f(d_j+r_f) - \sum_{F_3,F_7} c_f(d_i+d_j+r_f) - \sum_{F_{11},F_{15}} c_f(d_j+r_f) \right).
\end{aligned}
$$

By reordering the summation many terms cancel out and we obtain

$$
\begin{aligned}
I(\gamma,\pi) = &\sum_{f \in F_1 \cup F_{11}} (d_j - d_i)c_f(d_i + d_j + r_f) - d_j c_f(d_j + r_f) + d_i c_f(d_i + r_f) \\
&+ \sum_{f \in F_3 \cup F_9} (d_i - d_j)c_f(d_i + d_j + r_f) - d_i c_f(d_i + r_f) + d_j c_f(d_j + r_f),
\end{aligned}
$$

establishing the result. ◻

Using Lemma 3.3, we can derive a sufficient condition on the existence of an exact potential in a weighted congestion game.

Proposition 3.4. *Let $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ be a congestion model and $G(\mathcal{M})$ a corresponding weighted congestion game with demands $(d_i)_{i \in N}$. For each facility $f \in F$, we denote by $N^f = \{i \in N : (\exists x_i \in X_i : f \in x_i))\}$ the set of players potentially using f, and by $\mathcal{R}^f_{-\{i,j\}} = \left\{\sum_{k \in P} d_k : P \subseteq N^f \setminus \{i, j\}\right\}$ the set of possible residual demands by all players except i and j. If for all $f \in F$ and all $i, j \in N^f$*

$$
(d_j - d_i)c_f(d_i + d_j + r_f) - d_j c_f(d_j + r_f) + d_i c_f(d_i + r_f) = 0 \quad \forall r_f \in \mathcal{R}^f_{-\{i,j\}}, \tag{3}
$$

then G admits an exact potential.

Proof. Using the criterion of Monderer and Shapley, it is enough to prove that $I(\gamma, \pi) = 0$ for all $\gamma \in \Gamma(X)$. By Lemma 3.3, $I(\gamma, \pi)$ evaluates to

$$
\begin{aligned}
I(\gamma,\pi) = &\sum_{f \in F_1 \cup F_{11}} (d_j - d_i)c_f(d_i + d_j + r_f) - d_j c_f(d_j + r_f) + d_i c_f(d_i + r_f) \\
&+ \sum_{f \in F_3 \cup F_9} (d_i - d_j)c_f(d_i + d_j + r_f) - d_i c_f(d_i + r_f) + d_j c_f(d_j + r_f),
\end{aligned} \tag{4}
$$

for some $i, j \in N^f$ and $r_f \in \mathcal{R}^f_{-\{i,j\}}$. Using (3) each summand of (4) equals 0, establishing the result. ◻

It is a useful observation that we can write the condition of Proposition 3.4 as

$$\frac{c_f(d_i + d_j + r_f) - c_f(d_j + r_f)}{d_i} = \frac{c_f(d_j + r_f) - c_f(d_i + r_f)}{d_j - d_i} \tag{5}$$

for all $i, j \in N^f$ and $r_f \in \mathcal{R}^f_{-\{i,j\}}$. Thus, the difference quotients of c_f between the points $d_i + r_f$ and $d_j + r_f$ as well as $d_j + r_f$ and $d_i + d_j + r_f$ must have the same value. It follows easily that the above condition is satisfied if all demands are equal (this corresponds to unweighted congestion games, see Rosenthal's potential [24]). For *arbitrary* demands (weighted congestion games) and *affine* cost functions, one can check that the above condition is also satisfied, see the positive result of Fotakis et al. [12].

For a single weighted congestion game, the linearity condition on cost functions, however, is only sufficient but not necessary. In Example 3.5, we show that it is possible to construct a non-affine cost function that satisfies the condition of Proposition 3.4 for all 3 player games with demand vector $(1, 2, 5)$.

Example 3.5. *Let $\mathcal{M} = \big(N = \{1, 2, 3\}, X, F, (c_f)_{f\in F}\big)$ be an arbitrary congestion model with three players and let $G(\mathcal{M})$ be a corresponding weighted congestion game with demands $d_1 = 1, d_2 = 2, d_3 = 5$.*

We want to construct a non-linear cost function that gives rise to an exact potential in G. To this end, we consider an arbitrary 4-cycle γ. We apply Lemma 3.3 and obtain that $I(\gamma, \pi)$ evaluates to

$$\begin{aligned} I(\gamma, \pi) = &\sum_{f\in F_1\cup F_11} (d_j - d_i)c_f(d_i + d_j + r_f) - d_j c_f(d_j + r_f) + d_i c_f(d_i + r_f) \\ &+ \sum_{f\in F_3\cup F_9} (d_i - d_j)c_f(d_i + d_j + r_f) - d_i c_f(d_i + r_f) + d_j c_f(d_j + r_f), \end{aligned} \tag{6}$$

Regarding (6), only the following realizations of (d_i, d_j, r_f) are possible:

$$\begin{matrix} (1, 2, 0), & (1, 5, 0), & (2, 5, 0), \\ (1, 2, 5), & (1, 5, 2), & (2, 5, 1). \end{matrix} \tag{7}$$

Note that only realizations with $d_i < d_j$ are considered, the others are symmetric and, thus, omitted. Proposition 3.4 establishes that it is sufficient for the existence of an exact potential that in each cost function c_f, the values to the arguments shown in (7) lie on a straight line. It is easy to construct a non-linear cost function $c : \mathbb{R}_{>0} \to \mathbb{R}$ satisfying this property. An example of such a function is given in Fig. 1.

We derive that $I(\gamma, \pi) = 0$ for any 4-cycle γ in any such game regardless of the structure of the set of strategies.

There is, however, an important question left: Are there non-affine cost functions that give rise to an exact potential in *all* weighted congestion games? Under mild assumptions on feasible cost functions, we will give in Theorem 3.7 a negative answer to this question. First, we need the following lemma.

Lemma 3.6. *Let C be a set of functions and let $\mathcal{G}(C)$ be the set of all weighted congestion games with cost functions in C. Every $G \in \mathcal{G}(C)$ has an exact potential if and only if for all $c \in C$*

$$(x - y)\, c(x + y + z) - x\, c(x + z) + y\, c(y + z) = 0 \tag{8}$$

for all $x, y \in \mathbb{R}_{>0}$ and $z \in \mathbb{R}_{\geq 0}$.

$$c(\ell) = \begin{cases} 0 & \text{if } \ell \in (0,1] \\ 5\ell - 5 & \text{if } \ell \in (1,3] \\ -\ell + 13 & \text{if } \ell \in (3,5] \\ 2\ell - 2 & \text{if } \ell \in (5,6] \\ 10 & \text{if } \ell \in (6,\infty) \end{cases}$$

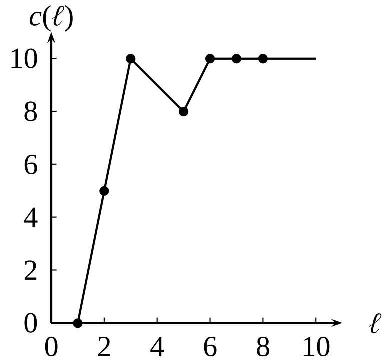

Figure 1: A non-linear cost function c_f that gives rise to an exact potential in all weighted congestion games with demands $d_1 = 1, d_2 = 3$ and $d_3 = 5$.

Proof. Suppose $G = (N, X, \pi) \in \mathcal{G}(C)$ is a weighted congestion game with cost functions in C. First, we will show that G has an exact potential. To this end, let $\gamma \in \Gamma(X)$ be an arbitrary simple closed path in X of length 4. $I(\gamma, \pi)$ evaluates to (1), which is zero using (8).

For the opposite direction suppose that there is a $\tilde{c} \in C$ that does not satisfy equation (8). This implies that there are $x, y \in \mathbb{R}_{>0}$ and $z \in \mathbb{R}_{\geq 0}$ such that

$$(x - y)\tilde{c}(x + y + z) - x\tilde{c}(x + z) + y\tilde{c}(y + z) \neq 0.$$

Now consider the congestion model $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ where $N = \{1, 2, 3\}$, $F = \{f, g, h\}$, $X_1 = \{\{f\}, \{g\}\}$, $X_2 = \{\{f\}, \{h\}\}$, $X_3 = \{f\}$, and $c_f = c_g = c_h = \tilde{c}$. Let $G = (N, X, \pi)$ be a corresponding weighted congestion game with demands $d_1 = y$, $d_2 = x$ and $d_3 = z$. We will investigate the value of $I(\gamma, \pi)$ for $\gamma = ((\{g\}, \{h\}, \{f\}), (\{f\}, \{h\}, \{f\}), (\{f\}, \{f\}, \{f\}), (\{g\}, \{f\}, \{f\}), (\{g\}, \{h\}, \{f\}))$. This value equals $(x - y)\tilde{c}(x + y + z) - x\tilde{c}(x + z) + y\tilde{c}(y + z) \neq 0$ implying that this game does not possess an exact potential function. $\square$

We will now solve the functional equation (8) in order to characterize all cost functions that guarantee an exact potential in all weighted congestion games. We require the following property: A function $c : \mathbb{R}_{>0} \to \mathbb{R}$ is *locally bounded*, if for every compact set $K \subset \mathbb{R}_{>0}$, $|c(x)| < M_K$ for all $x \in K$ and a constant $M_K \in \mathbb{R}_{>0}$ potentially depending on K.

Theorem 3.7. *Let C be a set of locally bounded functions and let $\mathcal{G}(C)$ be the set of weighted congestion games with cost functions in C. Then, every $G \in \mathcal{G}(C)$ admits an exact potential function if and only if C contains affine functions only, that is, every $c \in C$ can be written as $c(\ell) = a_c\,\ell + b_c$ for some $a_c, b_c \in \mathbb{R}$.*

Proof. It is straightforward to check that affine functions fulfill functional equation (8) and we may conclude that they give rise to an exact potential. We will prove the reverse direction in two steps.

In Step 1, we prove the following: Let c fulfill (8). Then, c is differentiable and $c'(x + z) = (c(x + z) - c(z))/x$ holds for all $x \in \mathbb{R}_{>0}$ and $z \in \mathbb{R}_{\geq 0}$.

First, we will show continuity of c on $\mathbb{R}_{>0}$. Let $x \in \mathbb{R}_{>0}$ and $z \in \mathbb{R}_{\geq 0}$ be arbitrary and let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_{>0}$ such that $y_n \overset{n\to\infty}{\longrightarrow} 0$ and both $y_n + z > 0$ and $y_n + x > 0$ for all $n \in \mathbb{N}$. Then, using (8) we get $x\,(c(x + z + y_n) - c(x + z)) = y_n\,(c(x + z + y_n) - c(z + y_n))$. As c is bounded on any compact set, the right hand side goes to 0 as n goes to infinity and hence $x \lim_{n\to\infty} c(x+z+y_n) - c(x+z) = 0$. This shows continuity in $x+z$.

Moreover, (8) implies that $x\,(c(x+z+y_n)-c(x+z))/y_n = c(x+z+y_n)-c(z+y_n)$. As c is continuous we know that the limits on the right hand side exist and, thus, $c'(x+z) = (c(x+z)-c(z))/x$ holds for all $x \in \mathbb{R}_{>0}$.

So c satisfies the differential equation $c'(x+z) = (c(x+z)-c(z))/x$. We will show in Step 2 that only affine functions solve this differential equation. To see this, we set $t = x+z$, which leads to the differential equation $c'(t) = (c(t)-c_0)\,/(t-t_0), t \in \mathbb{R}_{>0}$, where $c_0 = c(z)$ and $t_0 = z$ are constants. Standard calculus shows that for every initial value c_1 for the initial time $t_1 > t_0$, this ordinary linear differential equation admits a unique solution $c(t) = (t-t_0)\,C + c_0$, where $C = (c_1-c_0)/(t_1-t_0)$. □

3.2 Characterizing the Existence of a Weighted Potential

Our next aim is to determine whether weaker notions of potential functions will enrich the class of cost functions giving rise to a potential game. The idea of a weighted potential allows a player specific scaling of the private cost π_i by a strictly positive w_i. It is a useful observation that the existence of a weighted potential function is equivalent to the existence of a strictly positive-valued vector $w = (w_i)_{i\in N}$ such that the game G^w with private costs $\overline{\pi} := \bigtimes_{i\in N} \pi_i/w_i$ has an exact potential.

Using this equivalent formulation and Theorem 2.2 it follows that the existence of an exact potential function for the game $G^w = (N, X, \overline{\pi})$ is equivalent to $I(\gamma, \overline{\pi}) = 0$ for all $\gamma \in \Gamma(X)$ suggesting that G^w has an exact potential if and only if there are $w_i, w_j \in \mathbb{R}_{>0}$ such that

$$\left(\frac{d_i}{w_i} - \frac{d_j}{w_j}\right) c_f(d_i + d_j + r_f) = \frac{d_i}{w_i} c_f(d_i + r_f) - \frac{d_j}{w_j} c_f(d_j + r_f)$$

for all $i, j \in N$ and all $r_f \in \mathcal{R}^f_{-\{i,j\}}$. In particular it is necessary that either $c_f(d_i + d_j + r_f) = c_f(d_j + r_f) = c_f(d_i + r_f)$ or the value $\alpha(d_i, d_j)$ defined as

$$\alpha(d_i, d_j) = \frac{w_i}{w_j} = \frac{d_i}{d_j} \cdot \frac{c_f(d_i + d_j + r_f) - c_f(d_i + r_f)}{c_f(d_i + d_j + r_f) - c_f(d_j + r_f)} \tag{9}$$

is strictly positive and independent of both f and r_f. This observation leads us to the following lemma.

Lemma 3.8. *Let C be a set of functions. Let $\mathcal{G}(C)$ be the set of weighted congestion games with cost functions in C. Every $G \in \mathcal{G}(C)$ has a weighted potential if and only if for all $x, y \in \mathbb{R}_{>0}$, there exists an $\alpha(x, y) \in \mathbb{R}_{>0}$ such that*

$$\alpha(x, y) = \frac{x}{y} \cdot \frac{c(x + y + z) - c(x + z)}{c(x + y + z) - c(y + z)} \tag{10}$$

for all $z \in \mathbb{R}_{\geq 0}$ and non-constant $c \in C$.

Proof. Let $G = (N, X, \pi)$ be a weighted potential game in which all c_f are constant or satisfy equation (10). Let $(d_i)_{i\in N}, d_i \in \mathbb{R}_{>0}$, be an arbitrary vector of demands. We will show that this game possesses a weighted potential. For an arbitrary $c \in C$, we set $\alpha(d_i, d_j) = d_i/d_j \cdot (c(d_i + d_j) - c(d_i))/(c(d_i + d_j) - c(d_j))$ for $i, j \in N$ with $i \neq j$. The requirements of Lemma 3.8 imply that

$$\alpha(d_i, d_j) = \frac{d_i}{d_j} \cdot \frac{c(d_i + d_j + z) - c(d_i + z)}{c(d_i + d_j + z) - c(d_j + z)} \tag{11}$$

for all $z \in \mathbb{R}_{\geq 0}$ and $c \in C$. As $\alpha(d_i, d_j)$ is required to be strictly positive we may choose a vector of weights $(w_i)_{i\in N} \in \mathbb{R}_{>0}$ such that $\alpha(d_i, d_j) = w_i/w_j$ for all $i, j \in N$ with $i \neq j$. Using Monderer and Shapley's criterion we will show that the corresponding game $G^w = (N, X, \overline{\pi})$ has an exact potential. For this, we consider an arbitrary path $\gamma \in \Gamma(X)$. Without loss of generality only two players, say i and j, change their strategies in γ while the sum of the demands of all other players is equal to a facility-specific value r_f. We remark that facilities with constant cost function cancel out in the calculation of the integral. Analogously to the proof of Lemma 3.6, we get

$$
\begin{aligned}
I(\gamma, \overline{\pi}) = \\
&\sum_{f\in F_1,F_{11}} \left(\frac{d_j}{w_j} - \frac{d_i}{w_i}\right) c_f(d_i{+}d_j{+}r_f) - \frac{d_j}{w_j} c_f(d_j{+}r_f) + \frac{d_i}{w_i} c_f(d_i{+}r_f) \\
&+ \sum_{f\in F_3,F_9} \left(\frac{d_i}{w_i} - \frac{d_j}{w_j}\right) c_f(d_i{+}d_j{+}r_f) - \frac{d_i}{w_i} c_f(d_i{+}r_f) + \frac{d_j}{w_j} c_f(d_j{+}r_f).
\end{aligned}
$$

We multiply with w_i, use $\alpha(d_i, d_j) = w_i/w_j$ and obtain

$$
\begin{aligned}
w_i\, I(\gamma, \overline{\pi}) = \\
&\sum_{f\in F_1,F_{11}} \alpha(d_i, d_j) d_j \left(c_f(d_i{+}d_j{+}r_f) - c_f(d_j{+}r_f)\right) - d_i \left(c_f(d_i{+}d_j{+}r_f) + c_f(d_i{+}r_f)\right) \\
&+ \sum_{f\in F_3,F_9} -\alpha(d_i, d_j) d_j \left(c_f(d_i{+}d_j{+}r_f) - c_f(d_j{+}r_f)\right) + d_i \left(c_f(d_i{+}d_j{+}r_f) - c_f(d_i{+}r_f)\right).
\end{aligned}
$$

Using equation (11) shows that $I(\gamma, \overline{\pi}) = 0$ proving the first result.

To show the other direction, assume that the condition on C does not hold. In particular we can find two functions $\tilde{c}_1, \tilde{c}_2 \in C$, $z_1, z_2 \in \mathbb{R}_{\geq 0}$ and $x, y \in \mathbb{R}_{>0}$ such that at least two of the following four values are distinct

$$
\alpha(x, y)^{s,t} = \frac{x}{y} \cdot \frac{\tilde{c}_s(x + y + z_t) - \tilde{c}_s(x + z_t)}{\tilde{c}_s(x + y + z_t) - \tilde{c}_s(y + z_t)}, \qquad \text{where } s = 1, 2 \text{ and } t = 1, 2.
$$

We show the result for the case $\alpha(x, y)^{1,1} \neq \alpha(x, y)^{2,2}$ only. The other cases are similar. For this, we consider the congestion model $\mathcal{M} = (N, F, X, (c_f)_{f\in F})$, where $N = \{1, 2, 3\}$, $F = \{f_1, f_2, g_1, g_2, g_3\}$, $X_1 = \{\{g_1\}, \{f_1\}, \{f_2\}\}$, $X_2 = \{\{g_2\}, \{f_1\}, \{f_2\}\}$, $X_3 = \{\{g_3\}, \{f_1\}, \{f_2\}\}$, and $c_{f_1} = c_{g_1} = c_{g_2} = c_{g_3} = \tilde{c}_1, c_{f_2} = \tilde{c}_2$. Now we regard the games $G_1(\mathcal{M}) = (N, X, \pi)$ with demands $d_1 = x, d_2 = y, d_3 = z_1$ and $G_2(\mathcal{M}) = (N, X, \mu)$ with demands $d_1 = x, d_2 = y, d_3 = z_2$. As both games admit a weighted potential we can find two vectors of weights $(w_i)_{i\in N}$ and $(v_i)_{i\in N}$ for G_1 and G_2, respectively, such that $G_1^w = (N, X, \pi/w)$ and $G_2^v(N, X, \mu/v)$ admit exact potentials. To this end, we will apply the criterion of Monderer and Shapley to the path

$$
\gamma = ((\{g_1\}, \{g_2\}, \{g_3\}), (\{f_1\}, \{g_2\}, \{g_3\}), (\{f_1\}, \{f_1\}, \{g_3\}), (\{g_1\}, \{f_1\}, \{g_3\}), (\{g_1\}, \{g_2\}, \{g_3\}))
$$

in the games G_1^w and G_2^v. One verifies easily that the following equations hold

$$
I(\gamma, \pi/w) = \left(\frac{y}{w_2} - \frac{x}{w_1}\right) \tilde{c}_1(x + y) - \frac{y}{w_2} \tilde{c}_1(y) + \frac{x}{w_1} \tilde{c}_1(x) = 0, \tag{12}
$$

$$
I(\gamma, \mu/v) = \left(\frac{y}{v_2} - \frac{x}{v_1}\right) \tilde{c}_1(x + y) - \frac{y}{v_2} \tilde{c}_1(y) + \frac{x}{v_1} \tilde{c}_1(x) = 0. \tag{13}
$$

We define $\alpha(x, y) = w_1/w_2$ and $\xi(x, y) = v_1/v_2$. The equations (12) and (13) imply that

$$\alpha(x, y) = \xi(x, y) = \frac{x}{y} \cdot \frac{\tilde{c}_1(x+y) + \tilde{c}_1(x)}{\tilde{c}_1(x+y) - \tilde{c}_1(y)}. \tag{14}$$

Considering the path

$$\gamma_1 = ((\{g_1\}, \{g_2\}, \{f_1\}), (\{f_1\}, \{g_2\}, \{f_1\}), (\{f_1\}, \{f_1\}, \{f_1\}), (\{g_1\}, \{f_1\}, \{f_1\}), (\{g_1\}, \{g_2\}, \{f_1\}))$$

for G_1^w and the path

$$\gamma_2 = ((\{g_1\}, \{g_2\}, \{f_2\}), (\{f_2\}, \{g_2\}, \{f_2\}), (\{f_2\}, \{f_2\}, \{f_2\}), (\{g_1\}, \{f_2\}, \{f_2\}), (\{g_1\}, \{g_2\}, \{f_2\}))$$

for G_2^v, we can compute that

$$I(\gamma_1, \pi/w) = \left(\frac{y}{w_2} - \frac{x}{w_1}\right)\tilde{c}_1(x+y+z_1) - \frac{y}{w_2}\tilde{c}_1(y+z_1) + \frac{x}{w_1}\tilde{c}_1(x+z_1) = 0, \tag{15}$$

$$I(\gamma_2, \mu/v) = \left(\frac{y}{v_2} - \frac{x}{v_1}\right)\tilde{c}_2(x+y+z_2) - \frac{y}{v_2}\tilde{c}_2(y+z_2) + \frac{x}{v_1}\tilde{c}_2(x+z_2) = 0. \tag{16}$$

We derive from equations (15) and (16) that

$$\alpha(x, y) = \frac{x}{y} \cdot \frac{\tilde{c}_1(x+y+z_1) + \tilde{c}_1(x+z_1)}{\tilde{c}_1(x+y+z_1) - \tilde{c}_1(y+z_1)},$$
$$\xi(x, y) = \frac{x}{y} \cdot \frac{\tilde{c}_2(x+y+z_2) + \tilde{c}_2(x+z_2)}{\tilde{c}_2(x+y+z_2) - \tilde{c}_2(y+z_2)}.$$

Together with equation (14) this implies that

$$\alpha(x, y)^{1,1} = \frac{x}{y} \cdot \frac{\tilde{c}_1(x+y+z_1) - \tilde{c}_1(x+z_1)}{\tilde{c}_1(x+y+z_1) - \tilde{c}_1(x+z_1)} = \frac{x}{y} \cdot \frac{\tilde{c}_2(x+y+z_2) - \tilde{c}_2(x+z_2)}{\tilde{c}_2(x+y+z_2) - \tilde{c}_2(x+z_2)} = \xi(x, y) = \alpha(x, y)^{2,2},$$

which contradicts the assumption. ◻

Although condition 10 seems to be similar to the functional equation (8) characterizing the existence of an exact potential, it is not possible to proceed using differential equations. As $\alpha(x, y)$ need not be bounded it is not possible to prove continuity and differentiability of c. Instead, we will use the discrete counterpart of differential equations, that is, difference equations.

Theorem 3.9. *Let C be a set of continuous functions. Let $\mathcal{G}(C)$ be the set of weighted congestion games with cost functions in C. Then every $G \in \mathcal{G}(C)$ admits a weighted potential if and only if exactly one of the following cases holds:*

1. *C contains only affine functions,*
2. *C contains only exponential functions $c(\ell) = a_c\, e^{\phi \ell} + b_c$ for some $a_c, b_c, \phi \in \mathbb{R}$, where a_c and b_c may depend on c, while ϕ must be equal for every $c \in C$.*

Proof. First, we will prove that these functions guarantee the existence of a weighted potential in all such games. We have shown in Section 3.1 that affine cost functions c_f give rise to an exact potential. As every exact potential function is also a weighted potential for $w = (1, \ldots, 1)$, we may conclude that affine cost functions give rise to a weighted potential in weighted congestion games.

So let us check the case $c(\ell) = a_c\, e^{\phi\, \ell} + b_c$ for $\phi \neq 0$. It is easy to verify that

$$\alpha(x,y) = \frac{x}{y} \cdot \frac{a_c e^{\phi(x+y+z)} + b_c - a_c e^{\phi(x+z)} - b_c}{a_c e^{\phi(x+y+z)} + b_c - a_c e^{\phi(y+z)} - b_c} = \frac{x}{y} \cdot \frac{e^{\phi\,(x+y)} - e^{\phi(x)}}{e^{\phi\,(x+y)} - e^{\phi\,(y)}} > 0.$$

Note in particular that $\alpha(x, y)$ does neither depend on a_c, b_c, nor z. Thus, it is unambiguously defined and strictly positive. Theorem 3.8 then yields the result.

To show the opposite direction, we assume that the conditions on C do not hold but that every $G \in \mathcal{G}(C)$ admits a weighted potential.

First, suppose that there is a function $\tilde{c} \in C$ that is neither affine nor exponential. This implies that there are four points $p_1 < p_2 < p_3 < p_4$ following neither an exponential nor a affine law, that is, there are neither a, b and $\phi \in \mathbb{R}$ such that

$$\tilde{c}(p_1) = ae^{\phi p_1} + b, \qquad \ldots, \qquad \tilde{c}(p_4) = ae^{\phi p_4} + b,$$

nor are there s and $t \in \mathbb{R}$ such that

$$\tilde{c}(p_1) = sp_1 + t, \qquad \ldots, \qquad \tilde{c}(p_4) = sp_4 + t.$$

As $\tilde{c}$ is continuous, we may assume without loss of generality that the above conditions hold for rational $p_1, \ldots, p_4$ and we can write them as $p_1 = 2m_1/(2\,k), \ldots, p_4 = 2m_4/(2\,k)$ for some $m_1, m_2, m_3, m_4, k \in \mathbb{N}$.

We regard a congestion model $\mathcal{M} = (N = \{1, 2, 3\}, F, X, c)$ and a series of games $G_m(\mathcal{M}) = (N, X, \pi), 0 \leq m \leq 2\,m_4$. We set the demands of the players as $d_1 = 1/(2\,k)$, $d_2 = 2/(2\,k)$ and $d_3 = m/(2\,k)$. By assumption each game G_m admits a weighted potential. By Lemma 3.8 this implies that for each game

$$\alpha(d_1, d_2) = \frac{d_1}{d_2} \cdot \frac{\tilde{c}(d_1 + d_2 + d_3) - \tilde{c}(d_1 + d_3)}{\tilde{c}(d_1 + d_2 + d_3) - \tilde{c}(d_2 + d_3)} = \frac{d_1}{d_2} \cdot \frac{\tilde{c}(d_1 + d_2) - \tilde{c}(d_1)}{\tilde{c}(d_1 + d_2) - \tilde{c}(d_2)}.$$

In particular $\alpha(d_1, d_2)$ is the same for each game G_m. Now, we introduce $f_n = \tilde{c}(n/(2\,k))$ and consider the sequence $(f_n)_{n \in N}$. Thus, we can write

$$\alpha(d_1, d_2) = \frac{1}{2} \cdot \frac{f_{m+3} - f_{m+1}}{f_{m+3} - f_{m+2}}.$$

If $\alpha(d_1, d_2) = 1/2$, we conclude that $\tilde{c}$ is constant, which contradicts our assumption. So we may assume that $\alpha(d_1, d_2) \neq 1/2$ and we obtain

$$f_{m+2} - \frac{2\alpha(d_1, d_2)}{2\alpha(d_1, d_2) - 1} f_{m+1} + \frac{1}{2\alpha(d_1, d_2) - 1} f_m = 0. \tag{17}$$

Equation (17) defines a recursively defined sequence on $\{1, \ldots, 2m_4\}$.

The main result in [[5], Chapter 4] gives sufficient conditions on the uniqueness of the general solution of such sequences. First, we define the characteristic equation of a general second-order recurrence relation $a_{m+2} + \beta_2 a_{m+1} + \beta_1 a_m = 0$ as $x^2 + \beta_2 x + \beta_1 = 0$.

Now let x_1 and x_2 be the distinct and real roots of the characteristic equation. Then every general solution a_m of the recurrence relation is a linear combination of powers $(x_i)^m$ of the solutions $x_i, i = 1, 2$. In addition, if x is the double root of the characteristic equation, every general solution a_m of the recurrence relation is a linear combination of x^m and $m\, x^m$. In both cases, if two consecutive initial values a_k and a_{k+1} of the recurrence relation are known, a solution can be obtained by evaluating the two constants of the linear combination using the two initial values and the fact that this solution is unique.

The characteristic equation of the recurrence relation (17) equals

$$x^2 - \frac{2\alpha(d_1, d_2)}{2\alpha(d_1, d_2) - 1}x + \frac{1}{2\alpha(d_1, d_2) - 1} = (x - 1)\left(x - \frac{1}{2\alpha(d_1, d_2) - 1}\right).$$

So if $\alpha(d_1, d_2) \neq 1$, two different roots occur and f_m can be computed explicitely and uniquely for even m as

$$f_m = b \cdot 1^m + a \cdot \left(\frac{1}{2\alpha(d_1, d_2) - 1}\right)^m = b + a \cdot exp\left(m \ln\left(\frac{1}{2\alpha(k_1, k_2) - 1}\right)\right)$$

for some constants a and $b \in \mathbb{R}$. If $\alpha(k_1, k_2) = 1$, we can evaluate f_m as

$$f_m = t \cdot 1^m + m\, s \cdot 1^m = t + s\, m$$

for some constants $s, t \in \mathbb{R}$ showing that $\tilde{c}$ follows either an exponential or affine law on $p_1, \dots, p_4$. So it remains to show that neither affine and exponential functions nor exponential function with different exponents can occur simultaneously. Assume on the contrary that $\tilde{c}_1, \tilde{c}_2 \in C$ are two such functions and consider two players using both facilities f_1 and f_2 with cost functions $\tilde{c}_1$ and $\tilde{c}_2$, respectively. It is easy to show that these functions give rise to an ambiguously defined $\alpha(d_1, d_2)$ contradicting Theorem 3.8. □

3.3 Implications of Our Characterizations

It is natural to ask whether these results remain valid if additional restrictions on the set $\mathcal{G}(C)$ are made. A natural restriction is to assume that all players have an integral demand. As we used infinitesimally small demands in the proof of Lemma 3.6, our results for exact potentials do not apply directly to integer demands. With a slight variation of the proof of Theorem 3.9, where only the case $\alpha(\cdot, \cdot) = 1$ is considered, however, we still obtain the same result provided C contains only continuous functions.

Another natural restriction on $\mathcal{G}(C)$ are games with symmetric sets of strategies or games with a bounded number of players or facilities. Since the proofs of Lemma 3.6 and 3.8 and Theorems 3.7 and 3.9 rely on mild assumptions, we can strengthen our characterizations as follows.

Corollary 3.10. *Let C be a set of continuous functions. Let $\mathcal{G}(C)$ be the set of weighted congestion games with cost functions in C satisfying one or more of the following properties*

1. *Each game $G = (N, X, \pi) \in \mathcal{G}(C)$ has two (three) players.*

2. *Each game $G = (N, X, \pi) \in \mathcal{G}(C)$ has three (five) facilities.*

3. *For each game $G = (N, X, \pi) \in \mathcal{G}(C)$ and each player $i \in N$ the set of her strategies X_i contains a single facility only.*

4. *Each game $G = (N, X, \pi) \in \mathcal{G}(C)$ has symmetric strategies, that is $X_i = X_j$ for all $i, j \in N$.*

5. *In each game $G = (N, X, \pi) \in \mathcal{G}(C)$, the demands of all players are integral.*

Then, every $G = (N, X, \pi) \in \mathcal{G}(C)$ has an exact (a weighted) potential if and only if C contains only affine functions (only affine functions or only exponential functions as in Theorem 3.8).

Yet, we are able to deduce an interesting result concerning the existence of weighted potentials in weighted congestion games, where each facility can be chosen by at most two players. As we can set $z = 0$ in (10), the conditions of Lemma 3.8 are fulfilled by more than affine or exponential functions.

Theorem 3.11. *Let $m : \mathbb{R}_{>0} \to \mathbb{R}$ be a strictly monotonic function and let $C_m = \{a\, m(x) + b : a, b \in \mathbb{R}\}$. Then every two-player weighted congestion game $G \in \mathcal{G}^2(C_m)$ admits a weighted potential.*

Proof. By adapting the proof of Lemma 3.8 for two-player games, we establishes the following lemma.

Lemma 3.12. *Let C be a set of functions. Let $\mathcal{G}^2(C)$ be the set of two-player weighted congestion games with cost functions in C. Every $G \in \mathcal{G}^2(C)$ has a weighted potential if and only if for all $x, y \in \mathbb{R}_{>0}$ there exists an $\alpha(x, y) \in \mathbb{R}_{>0}$ such that*

$$\alpha(x, y) = \frac{x}{y} \cdot \frac{c(x + y) - c(x)}{c(x + y) - c(y)} \tag{18}$$

for all non-constant $c \in C$.

Let $\tilde{c} \in C$ be arbitrary. By the definition of C, we can write $\tilde{c}(\ell) = a_{\tilde{c}}\, m(x) + b_{\tilde{c}}$ for some $a_{\tilde{c}}, b_{\tilde{c}} \in \mathbb{R}$. If $a_{\tilde{c}} = 0$, the function $\tilde{c}$ is constant and thus fulfills the requirements of Lemma 3.12. If $a_{\tilde{c}} \neq 0$, it is easy to check that

$$\begin{aligned} \alpha &= \frac{d_1}{d_2} \cdot \frac{\tilde{c}(d_1 + d_2) - \tilde{c}(d_1)}{\tilde{c}(d_1 + d_2) - \tilde{c}(d_2)} = \frac{d_1}{d_2} \cdot \frac{a_{\tilde{c}}\, m(d_1 + d_2) + b_{\tilde{c}} - (a_{\tilde{c}}\, m(d_1) + b_{\tilde{c}})}{a_{\tilde{c}}\, m(d_1 + d_2) + b_{\tilde{c}} - (a_{\tilde{c}}\, m(d_2) + b_{\tilde{c}})} \\ &= \frac{d_1}{d_2} \cdot \frac{m(d_1 + d_2) - m(d_1)}{m(d_1 + d_2) - m(d_2)} > 0 \end{aligned}$$

for all $\tilde{c} \in C$ and hence the conditions of Lemma 3.12 are fulfilled implying the existence of a weighted potential. □

This result generalizes a result of Anshelevich et al. in [3], who showed that a weighted congestion game with two players and $c_f(\ell) = b_f / \ell$ for a constant $b_f \in \mathbb{R}_{>0}$ has a potential. Moreover, this result shows that the characterization of Corollary 3.10 is tight in the sense that weighted congestion games with two players admit a weighted potential even if cost functions are neither affine nor exponential.

4 Extensions of the Model

In the last section, we developed a new technique to characterize the set of functions that give rise to a potential in weighted congestion games. In this section, we will introduce two generalizations of weighted congestion games and investigate the set of cost functions that assure the existence of potential functions.

Definition 4.1 (Facility-dependent demands)**.** Let $\mathcal{M} = (N, F, X, (c_f)_{f\in F})$ be a congestion model and let $\left(d_{i,f}\right)_{i\in N, f\in F}$ be a matrix of facility-dependent demands. The corresponding *weighted congestion game with facility-dependent demands* is the strategic game $G(\mathcal{M}) = (N, X, \pi)$, where π is defined as $\pi = \bigtimes_{i\in N} \pi_i$, $\pi_i(x) = \sum_{f\in x_i} d_{i,f}\, c_f(\ell_f(x))$ and $\ell_f(x) = \sum_{j\in N: f\in x_j} d_{j,f}$.

Restricting the strategy sets to singletons, we obtain scheduling games. In a scheduling game, players are jobs that have machine-dependent demands and can be scheduled on a set of admissible machines (restricted scheduling on unrelated machines). In contrast to the classical approach, where each job strives to minimize its makespan, we consider a different private cost function: Machines charge a price per unit given by a load-dependent cost function c_f and each job minimizes its cost defined as the price of the chosen machine multiplied with its machine-dependent demand.

Theorem 4.2. *Let C be a set of continuous functions and let $\mathcal{G}^{fd}(C)$ be the set of weighted congestion games with facility-dependent demands and cost functions in C. Then, every $G \in \mathcal{G}^{fd}(C)$ admits a weighted potential if and only if C contains only affine functions, that is, every $c \in C$ can be written as $c(\ell) = a_c\,\ell + b_c$ for some $a_c, b_c \in \mathbb{R}$. For a game G with affine cost functions, the potential function is given by $P(x) = \sum_{i\in N} \sum_{f\in x_i} c_f\left(\sum_{j\in\{1,\dots,i\}: f\in x_j} d_{j,f}\right) d_{i,f}$.*

Proof. For any set of functions C, the set $\mathcal{G}(C)$ of weighted congestion games with cost functions in C is contained in the set of weighted congestion games with facility-dependent demands. Thus, we can restrict C to the set of affine functions or exponential functions as in Theorem 3.9. By varying the demands between two facilities with exponential costs it is easy to verify that the weights w_i and w_j are ambiguously defined and we may derive that C does not contain any exponential functions. We thus proceed by showing that $P(x)$ is an exact potential for affine costs.

Cost functions are affine, that is, $c_f(\ell) = a_f\ell + b_f$, $a_f, b_f \in \mathbb{R}$. We define the function $c_f^{\leq i}(x) := c_f\Big(\sum_{j\in\{1,\dots,i\}: f\in x_j} d_j^f\Big)$ and rewrite $P(x)$ as $P(x) = \sum_{i\in N} P_i(x)$, where $P_i(x) = \sum_{f\in x_i} c_f^{\leq i}(x) d_i^f$. Let $G = (N, X, \pi)$ be an arbitrary weighted congestion game with facility-dependent demands and let $x, y \in X$ be two strategy profiles such that $x = (x_k, x_{-k})$ and $y = (y_k, y_{-k})$ with $x_{-k} = y_{-k}$ for some $x_k, y_k \in X_k$ and $x_{-k} \in X_{-k}$. We notice that $P_i(x) = P_i(y)$ for all $i < k$. Now consider a player $i > k$. When computing $P_i(x) - P_i(y)$, we observe that all costs corresponding to facilities not contained in $x_k \cup y_k$ cancel out. For each facility $f \in (x_i \cap x_k) \setminus y_k$, we see that $c_f^{\leq i}(x) - c_f^{\leq i}(y) = a_f\, d_k^f$. Analogously, for each facility $f \in (x_i \cap y_k) \setminus x_k$, it holds that $c_f^{\leq i}(x) - c_f^{\leq i}(y) = -a_f d_k^f$. For each facility $f \in x_i \cap x_k \cap y_k$, we have $c_f^{\leq i}(x) = c_f^{\leq i}(y)$. Hence,

$$P_i(x) - P_i(y) = \sum_{f\in x_i\cap x_k} a_f\, d_k^f\, d_i^f - \sum_{f\in x_i\cap y_k} a_f\, d_k^f\, d_i^f.$$

Moreover, we can calculate straightforwardly that

$$P_k(x) - P_k(y) = \sum_{f\in x_k} c_f\Big(\sum_{j\in\{1,\dots,k\}: f\in x_j} d_j^f\Big) d_k^f - \sum_{f\in y_k} c_f\Big(\sum_{j\in\{1,\dots,k\}: f\in y_j} d_j^f\Big) d_k^f.$$

We thus obtain

$$
\begin{aligned}
&P(x) - P(y) \\
&\quad = \sum_{i \in N} P_i(x) - \sum_{i \in N} P_i(y) \\
&\quad = \sum_{i>k}^{n} \Big(\sum_{f \in x_i \cap x_k} a_f d_k^f d_i^f - \sum_{f \in y_i \cap y_k} a_f d_k^f d_i^f \Big) \\
&\qquad + \sum_{f \in x_k} a_f \Big(\sum_{j \in \{1,\dots,k\}: f \in x_j} d_j^f \Big) d_k^f - \sum_{f \in y_k} a_f \Big(\sum_{j \in \{1,\dots,k\}: f \in y_j} d_j^f \Big) d_k^f + d_k^f \sum_{f \in x_k} b_f - d_k^f \sum_{f \in y_k} b_f \\
&\quad = \sum_{f \in x_k} a_f \Big(\sum_{j \in \{1,\dots,n\}: f \in x_j} d_j^f \Big) d_k^f - \sum_{f \in y_k} a_f \Big(\sum_{j \in \{1,\dots,n\}: f \in y_j} d_j^f \Big) d_k^f + d_k^f \sum_{f \in x_k} b_f - d_k^f \sum_{f \in y_k} b_f \\
&\quad = \pi_k(x) - \pi_k(y).
\end{aligned}
$$

□

We will now introduce an extension to weighted congestion games allowing players to also choose their demand.

Definition 4.3 (Elastic demands)**.** Let $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ be a congestion model. Together with $D = \bigtimes_{i \in N} D_i$, where $D_i \subset \mathbb{R}_{>0}$ are compact for all $i \in N$, we define the *weighted congestion game with elastic demands* as the strategic game $G(\mathcal{M}) = (N, \overline{X}, \pi)$ with $\overline{X} := (X, D)$, $\pi = \bigtimes_{i \in N} \pi_i$, and $\pi_i(\overline{x}) = \sum_{f \in x_i} d_i c_f(\ell_f(\overline{x}))$ and $\ell_f(\overline{x}) = \sum_{j \in N: f \in x_j} d_j$.

In our definition of weighted congestion games with elastic demands, we explicitly allow for positive and negative, and for increasing and decreasing cost functions. Thus, an increase in the demand may increase or decrease the player's private cost. Note that in weighted congestion games with elastic demands, the strategy sets are topological spaces and are in general infinite. By restricting the sets D_i to singletons $D_i = \{d_i\}$, $i \in N$, we obtain weighted congestion games as a special case of weighted congestion games with elastic demands. The proof of the following result is omitted as it is similar to the case of facility-dependent demands.

Theorem 4.4. *Let C be a set of continuous functions and let $\mathcal{G}^e(C)$ be the set of weighted congestion games with elastic demands and cost functions in C. Then, every $G \in \mathcal{G}^e(C)$ admits a weighted potential function if and only if C contains only affine functions. For a game G with affine cost functions, the potential function is given by the function $P(\overline{x}) = \sum_{i \in N} \sum_{f \in x_i} c_f \left(\sum_{j \in \{1,\dots,i\}: f \in x_j} d_j \right) d_i$.*

As an immediate consequence, we obtain the existence of a PNE if cost functions are affine. Note that the existence of a potential is not sufficient for proving existence of a PNE as we are considering infinite games. However, as $\overline{X}$ is compact and P is continuous, P has a minimum $\overline{x}^* \in \overline{X}$ and $\overline{x}^*$ is a PNE.

Corollary 4.5. *Let C be a set of affine functions and let $\mathcal{G}^e(C)$ be the set of weighted congestion games with elastic demands and cost functions in C. Then, every $G \in \mathcal{G}^e(C)$ admits a PNE.*

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